Lesson 1

ALGEBRA OF MATRICES

OBJECTIVES

At the end of this session, you will be able to understand:

- Introduction of Matrices
- Definition of Matrix
- Special Types of Matrices
- Operations of Matrices
- Determinant of a Matrix
- Difference between a Matrix and Determinant
- Adjoint of a Matrix
- Inverse of a Matrix

INTRODUCTION: A French mathematician 'CAYLEY' discovered the method of matrices in the year 1860 Matrices have been found to be powerful and useful tool for solving many problems involving electrical circuits, linear equations, mechanics, system of differential equations, astronomy and aerodynamics. So matrices have wide applications in modern mathematics, engineering and technology. Therefore, it is necessary for the young engineers to be familiar with the concept of matrix.

DEFINITION: A system of *mn* numbers (real or complex) arranged in the form of a rectangular array of m-rows (horizontal lines) and n- columns (vertical lines) is called a matrix of order $m \times n$ and is written as $m \times n$ matrix (read as m by n matrix). Each of the *mn* numbers is called an element of the matrix.

A matrix is also denoted by a single capital letter.

$$\text{Thus } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

When it has m rows and n columns,

To show any particular element of a matrix, the elements are denoted by a letter followed by double suffixes, which respectively specify the rows and columns. Thus a_{ij} is the element in the i^{th} row and j^{th} column of A. In this notation, the matrix A is denoted by $|a_{ij}|$ or (a_{ij}).

A matrix should be taken as a single entity with a number of components, rather than a collection of numbers. The stress at a point inside an elastic solid has nine components and it can be expressed as a 3 × 3 matrix. Unlike a determinants matrix does not reduce to a single number.

SPECIAL TYPES OF MATRICES:

(i) **Row Matrix:** A matrix having only row is called a row matrix or a row vector. The given matrix

$$[a_{11} \;\; a_{12} \;\; a_{13} \; \ldots\ldots\ldots\ldots a_{1n}]$$

is said to be a row matrix of order $l \times n$.

(ii) **Column Matrix:** A matrix having only one column is called a column matrix or a column vector. The given matrix

$$\begin{bmatrix} a_{11} \\ a_{21} \\ : \\ : \\ a_{m1} \end{bmatrix}$$

is said to be a column matrix of order $m \times l$

(iii) **Rectangular Matrix:** Any m by n matrix where $m \neq n$ is called a rectangular matrix,

e.g $\begin{bmatrix} 3 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix}$ is a rectangular matrix of order 2x3

(iv) **Square Matrix:** If the numbers of rows and columns in a matrix are equal (m = n), then the matrix is called a square matrix of order $n \times n$.

The diagonal containing the elements a_{11}, a_{22}, a_{33}, ……a_{nn} is called principal or leading diagonal.

(v) **Diagonal Matrix:** In a square matrix , if all the elements except those of the principal diagonal are zero, then it is called a diagonal matrix, e.g.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix of order 3

The above matrix can be written as diag. (1, 2, 3) and the elements 1, 2, 3 are its diagonal elements.

(vi) **Scalar Matrix:** A diagonal matrix having all the same elements along the principal diagonal is called a scalar matrix, e.g.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \text{is a scalar matrix of order 3.}$$

(vii) **Unit Matrix or Identity Matrix:** A square matrix whose all diagonal elements are all equal to unity is called a unit matrix or Identity matrix e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{is a identity matrix of order 3.}$$

It is generally denoted by I or I_n, where n is the order of the square matrix.

(viii) **Null Matrix or Zero Matrix:** A matrix whose all elements are zero is called a null matrix or zero matrix and denoted by O e.g.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{is a null matrix of order } 3 \times 3$$

(ix) **Triangular Matrix:** A square matrix whose elements either below or above the leading diagonal, all are zero is called a triangular matrix. When all the elements of a square matrix below the leading diagonal are zero, it is called an upper triangular matrix and if all the elements of a square matrix above the leading diagonal are zero, it is called a lower triangular matrix.

Thus A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} and\ A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ are upper and lower triangular metrics.

(x) **Transpose of Matrix:** If the rows and columns of a matrix a interchanged, then obtained new matrix is called the transpose of matrix and is denoted by A^T or A'. Thus if

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 0 \\ 3 & -1 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & -3 & 3 \\ -3 & 0 & -1 \end{bmatrix}$$

It is clear that if A = $|a_{ij}|_{m\times n}$ then $A^T = |a_{ji}|_{m\times n}$ i.e. *j-i*th element of A^T is equal to the *i-j*th element of A.

PROPERTIES OF TRANSPOSES:

If A^T and B^T are the transpose matrices of A and B respectively, then

(a) $(A^T)^T = A$
(b) $(A+B)^T = A^T + B^T$, A and B being comparable matrices,
(c) $(kA)^T = kA^T$, k being any scalar,
(d) $(AB)^T = B^T A^T$, A and B being conformable for multiplication.

(xi) **Singular and Non-Singular Matrices:** A square matrix A is said to be singular and non-singular matrices according as $|A| = 0$ and $|A| \neq 0$ I e, if determinant of A = 0, then A is said to be a singular matrix and if determinant of $A \neq 0,$ then A is said to be a non-singular matrix.

(xii) **Symmetric Matrix:** A square matrix is said to be symmetric matrix if it is equal to its transpose i.e. if $A = A^T$ or $a_{ij} = a_{ji}$ for all i & j, e.g.

$$A = \begin{bmatrix} 3 & 5 & 4 \\ 5 & 2 & 0 \\ 4 & 0 & 4 \end{bmatrix} \text{ is a symmetrix matrix of order } 3\times 3$$

(xiii) **Skew-Symmetric Matrix:** A square matrix is said to be skew-symmetric matrix if $A^T = -A$ or $a_{ij} = -a_{ji}$, for all i & j, so that a_{ii} = o for all i, e.g.

$$A = \begin{bmatrix} 0 & 2 & 3+4i \\ -2 & 0 & -7 \\ -3-4i & 7 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix} \text{ are skew-symmetric matrix of order } 3\times 3.$$

(xiv) **Submatrices of a Matrix:** Any matrix obtained by deleting rows or columns, or both of a matrix A, is called a submatrix of A e.g. the matrix

$$A_1 = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} \text{ is a submatrix of the matrix } A = \begin{bmatrix} 3 & 4 & 5 & 8 \\ 6 & 7 & 9 & 10 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

as it can be obtained from A by omitting the second row and the fourth column.

(xv) **Equality of Two Matrices:** Two matrices A = $\lfloor a_{ij} \rfloor$ and B = $\lfloor b_{ij} \rfloor$ are said to be equal if and only if

(i) they are of the same order and
(ii) each element of A is equal to the corresponding element of B, i.e, $a_{ij} = b_{ij}$ for pair of subscripts i & j.

OPERATIONS OF MATRICES:

(i) **Addition of Matrices:** Two matrices A and B are conformable for addition if they are of same order, then their sum A + B is defined as matrix whose each element is the sum of the corresponding elements of A and B. Thus if A=$\lfloor a_{ij} \rfloor_{m\times n}$, B = $\lfloor b_{ij} \rfloor_{m\times n}$ then, A + B = $\lfloor a_{ij} + b_{ij} \rfloor_{m\times n}$.

More clearly we can say that if

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & . & a_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & . & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & . & . & . & b_{1n} \\ b_{21} & b_{22} & . & . & . & b_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{m1} & b_{m2} & . & . & . & b_{mn} \end{bmatrix}$$

$$\text{Then } A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & . & . & . & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & . & . & . & a_{2n} + b_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & . & . & . & a_{mn} + b_{mn} \end{bmatrix}$$

For example,

Let $A = \begin{bmatrix} 3 & 2 \\ 4 & -3 \end{bmatrix}_{2x2}$ and $B = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}_{2x2}$

Then $A + B = \begin{bmatrix} 4 & 0 \\ 7 & -1 \end{bmatrix}_{2x2}$

(ii) **Subtraction of Matrix:** If A and B are two $m \times n$ matrices of same order, then the difference A – B may be written as the sum of the matrices A and (-B) i.e.,

$$A - B = A + (-B)$$

Note: Two matrices of different orders cannot be added or subtracted.

PROPERTIES OF MATRIX ADDITION:

1. **Commutative law for matrix addition**: If A and B are two $m \times n$ matrices, then
$$A + B = B + A$$

2. **Associative law for matrix addition**: If A, B, C are any three matrices each of the same order $m \times n$, then
$$(A+B)+C = A+(B+C).$$

3 **Existence of the additive identity**: If O be the null matrix of the same order as A, then $A+O = O+A$.

4 **Existence of the additive inverse:** If A is any given matrix of order $m \times n$ then there exists a matrix –A which is the additive inverse of A
$$\therefore A + (-A) = (-A) + A = 0.$$

5 **Cancellation law for matrix addition**: If A, B, C are comparable matrices, then the relation $A + B = A + C$ holds if and only if B = C.

(iii) **Multiplication of Matrix by a Scalar**: IF A is a matrix of the order $m \times n$ and α is a scalar, then the matrix obtained by multiplying each element of matrix A by α is called the scalar multiple of A by α and denoted as α A or Aα . Thus if A = $\left|a_{ij}\right|_{m\times n}$ then α A = Aα = $\left|\alpha a_{ij}\right|_{m\times n}$.

Example: Evaluate α A, where α = 3, A = $\begin{bmatrix} 3 & 2 & 1 \\ 4 & -3 & 1 \end{bmatrix}_{2\times 3}$

Solution: We have 3A = $3\begin{bmatrix} 3 & 2 & 1 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 3 \\ 12 & -9 & 3 \end{bmatrix}$

Example: Evaluate 3A +4B, where

$$A = \begin{bmatrix} 3 & -4 & 6 \\ 5 & 1 & 7 \end{bmatrix} and\ B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

Solution: We have 3A = $\begin{bmatrix} 9 & -12 & 18 \\ 15 & 3 & 21 \end{bmatrix}$

And 4B = $\begin{bmatrix} 4 & 0 & 4 \\ 8 & 0 & 12 \end{bmatrix}$

$$\therefore 3A + 4B = \begin{bmatrix} 13 & -12 & 22 \\ 23 & 3 & 33 \end{bmatrix}$$

(iv) **Multiplication of Matrices**: Two matrices can be multiplied only when the number of columns in the first is equal to the number of rows in the second. Such matrices are said to be conformable. If A and B are two matrices, then their product AB is defined only if the number of columns of A is equal to number of rows of B. If A = $\lfloor a_{ij} \rfloor_{m\times n}$ $and B = \lfloor b_{jk} \rfloor_{m\times p}$ are conformable for multiplication, then the m×p matrix C = $\lfloor c_{jk} \rfloor_{m\times p}$ such that

$$c_{jk} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

is called the product of the matrices A and B in that order and we write C =A B.

In the product AB, A is called the pre-factor and the matrix B is called the post-factor.

For example, if A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$ $and B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2}$

Then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}_{3\times 2}$$

In general, if

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix}_{m\times n} \quad and \quad B = \begin{bmatrix} b_{11} & b_{12} & . & . & b_{1n} \\ b_{21} & b_{22} & . & . & b_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{m1} & b_{m2} & . & . & b_{mn} \\ & & & & \end{bmatrix}_{n\times p}$$

be two conformable matrices, then their product is defined as the m×p matrix

$$AB = \begin{bmatrix} c_{11} & c_{12} & . & . & c_{1p} \\ c_{21} & c_{22} & . & . & c_{2p} \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{m1} & c_{m2} & . & . & c_{mp} \end{bmatrix}$$

PROPERTIES OF MATRIX MULTIPLICATION:

1. **Matrix Multiplication is Associative Law**. If A, B, C be three matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively, then

   A(BC) = (AB)C.

2. **Matrix Multiplication is Distributive w.r.t Addition of Matrices**: If A, B, and C are any of three matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively, then

   A(B+C) = AB + AC.

3. **Matrix Multiplication is not Commutative:** In general $AB \neq BA$

4. **Matrix Multiplication by a Null Matrix:** If A is a matrix of order $m \times n$ and O is a null matrix of order $n \times m$, then

   AO = O =OA.

5. **Matrix Multiplication by a Unit Matrix:** If A is a square matrix of order n and I is a unit matrix of the same order, then

   IA =AI =A

DETERMINANTS:

(i) **Determinants of order 2**: Let $a_{11'}$ $a_{12'}$ a_{21} a_{22} be any four numbers (real or complex). The symbol

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

represents the number $a_{11}\, a_{12} - a_{21}\, a_{12}$ and it is called a determinant of order 2. "The value of a determinants of order 2 is equal to the product of the elements along the principal diagonal minus the product of the rest diagonal elements". Thus

For example, $\begin{vmatrix} -3 & 6 \\ -4 & 7 \end{vmatrix} = (-3).(7) - (-4).(6) = -21 + 24 = 3$

(ii) **Determinant of order 3**: The symbol

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is called a determinant of order 3 and its value is given by

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{23} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \quad \ldots\ldots(1)$$

This is called expansion of the determinant along the first row.

There are three rows and three columns in a determinant of order 3. It has got 3x3 i.e., 9 elements, we can find the value of a determinant of order 3 by expanding if along any of its row or along any of its columns. For example, if we expand Δ along the first column, then

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a1_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \quad \ldots\ldots(2)$$

We observe that (1) and (2) are same elements.

In these expansions the element a_{ij} is multiplied by $(-1)^{I+j}$ to fix the sign of a_{ij}

Determinant of a Matrix: Let A be any square matrix. The determinant formed by the elements of A is said to be the determinant of matrix A and this is denoted by det. A or A. Since in a determinant the number of rows is equal to the number of columns, therefore only square matrices can have determinants. Thus if

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 2 & 1 & 2 \end{bmatrix}$$

The det. A can be find as

$$|A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix}$$

$= 1(-2-3) - 3(4-6) + 4(2+2) = -5 + 6 + 16 = 17.$

DIFFERENCE BETWEEN A MATRIX AND A DETERMINANT:

(i) A matrix can not be reduced to a single number. But a determinant can be reduced to a single number.
(ii) The number of rows may or may not be equal to number of column in a matrix while in a determinant the number of rows is equal to the number of columns.
(iii) An interchange of rows (or columns) in matrix gives rise to a different matrix. But an interchange of rows (or column) in a determinant gives rise to the same determinant but with positive or negative sign.

ADJOINT OR ADJUGATE OF A MATRIX: Let A = $|a_{ij}|_{m\times n}$ be any square matrix. The transpose B^T of the matrix $B = |a_{ij}|$, where A_{ij} denotes the co-factor of the element a_{ij} in the determinant A, is called the adjoint of a matrix A and is denoted by adj. A Thus if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then}$$

$$|A| = a_{11}(a_{22}a_{23} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$
$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13},$$

Where $A_{11}, A_{12} and A_{13}$ are called the co-factor of first row of a_{11}, a_{12}, and a_{13} respectively. Similarly, co-factor of second row are A_{21}, A_{22} and A_{23}, co-factor of third row are A_{31},

A_{32} and A_{33}, then co-factor matrix B = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$\therefore adj.A =$ Transpose of cofactor matrix B $= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

PROPERTIES OF ADJOINT MATRIX:

1. If A is square matrix, then A(adj.A) = (adj.A)A = $|A|I$, where I is the n×n unit matrix

2. The adjoint of an identity matrix is the identity matrix.

3. The adjoint of a diagonal matrix is diagonal matrix.

4. The adjoint of a symmetric matrix is a symmetric matrix.

5. The adjoint of a scalar matrix is a scalar matrix.

Example: Obtain the adjoint of matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$$

Solution.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = (9-2) = 7, A_{12} = (-1)^{1+2}\begin{vmatrix} -5 & 1 \\ -3 & 3 \end{vmatrix} = -(-15+3) = 12.$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -5 & 3 \\ -3 & 2 \end{vmatrix} = -(-3-6) = 9, A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -(-3-6\_ = 9.$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = (-1-9) = -10, A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = -(2+15) = -17.$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -1 \\ -5 & 3 \end{vmatrix} = (6-5) = I.$$

Co-factors of matrix is given by

$$B = \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \text{adj. A} = B^T = \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

INVERSE OR RECIPROCAL OF A MATRIX: if A is any square matrix of order n and if another matrix B exists of same order such that AB = BA = I_n I_n is the unit matrix of order n then B is called the inverse or reciprocal matrix of A. Inverse of A is denoted by A^{-1}

We have already seen that

$$A(\text{adj. } A) = (\text{adj. } A)\,A = |A|\,I, \text{ or } A.\frac{adj.A}{|A|} = \frac{adj.A}{|A|}\,A = I,$$

Hence, we get $A^{-1} = \frac{adj.A}{|A|}$, provided $|A| \neq 0$.

PROPERTIES OF INVERSE MATRICES:

1. The inverse of a matrix is always unique.
2. If A and B are non-singular matrices of order n x n, then

   $(AB)^{-1} = B^{-1} A^{-1}$

3. If A is non-singular matrix, then

   $(A^{-1})^{-1} = A.$

4. If A is non-singular matrix, then

   $(A^{-1})^{T} = (A^{t})^{-1}$

Example: Obtain A^T, adj. A and A^{-1}, when

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

Solution: By changing rows into columns and columns into rows, we can get.

$$A^T = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & -6 \\ -1 & 5 & -7 \end{bmatrix}$$

To obtain adj. A, now consider $|A|$ of the matrix A

$$|A| = 1\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} = 1.(2) - 0 + (-1)(-18) = 20$$

Co-factor of the first row are

$$A_{11} = \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = -28 + 30 = 2, A_{12} = -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = 21.$$

$$A_{13} = \begin{vmatrix} 3 & 4 \\ 0 & 6 \end{vmatrix} = -18.$$

Similarly co-factors is second row are $A_{21} = 6$, $A_{22} = -7$, $A_{23} = 6$ and of the third row $A_{31} = 4$, $A_{32} = -8$, $A_{33} = 4$, then

$$\text{Co-factor matrix } B = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$$

$$\text{Hence adj. } A = B^T = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$|A| = 20,$$

Further $\therefore A^{-1} = \dfrac{adj.A}{|A|}$

$$= \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{21}{20} & \frac{-7}{20} & \frac{-2}{5} \\ \frac{-9}{10} & \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

ADDITIONAL PROBLEMS

1. If $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, find the product of AB and BA and show that $AB \neq BA$.

2. For what value of x, the matrix $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & 1-x \end{bmatrix}$ is singular?

3. Show that $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \times \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, when θ *and* ϕ differ by and odd multiple of $\frac{\pi}{2}$.

4. Show that inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$ *is* $\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$, where ω is a complex cube root of unity.

5. If $A = \begin{bmatrix} 0 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 0 \end{bmatrix}$ show that $I + A = (I - A)\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$